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Order Convergence and Order Topology on a Poset

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Relationships between order convergence and order convergence topology in a poset and its MacNeille completion are studied.

1. INTRODUCTION

A net $\{x_\alpha\}_{\alpha\in\mathscr{E}}$ in a partially ordered set $(P, \leq)$ is said to be *order convergent* (Birkhoff, 1973) to a point $x \in P$ ($x_\alpha \xrightarrow{(o)} x$) iff there are two nets, an increasing net $\{u_\alpha\}_{\alpha\in\mathscr{E}}$ and a decreasing net $\{v_\alpha\}_{\alpha\in\mathscr{E}}$, such that $u_\alpha \leq x_\alpha \leq v_\alpha$ for every $\alpha \in \mathscr{E}$ and

$$\bigvee_{\alpha\in\mathscr{E}} u_\alpha = \bigwedge_{\alpha\in\mathscr{E}} v_\alpha = x$$

The *order topology* on $(P, \leq)$ is defined in Birkhoff (1973) by order convergence as follows: A set $C \subset P$ is said to be closed iff it contains the limits of all order-convergent nets in C. It is the finest topology on P which preserves the order convergence. It follows that every order-convergent net in P is convergent with respect to the order topology ($x_\alpha \xrightarrow{\tau_o} x$).

Kirchheimová (1990) disproved a statement of Birkhoff's that in an arbitrary poset P, order convergence of nets is inherited from the MacNeille completion of P (Birkhoff, 1973, Chap. X, par. 9). In addition, in that monograph no distinction is made between order convergence of nets and their convergence in the order topology on a poset. Thus, it appears reasonable to investigate the relationship between the order convergence and the convergence with respect to the order topology as well as the relationship between both the order convergence and the order topology in a poset P and in its MacNeille completion.

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From the definitions, it immediately follows that

$$x_\alpha \overset{(o)}{\to} x \Rightarrow x_\alpha \overset{(\hat{o})}{\to} x \qquad \text{for all} \quad x_\alpha, x, \in P$$

$$x_\alpha \overset{\tau_o}{\to} x \Rightarrow x_\alpha \overset{\hat{\tau}_o}{\to} x \qquad \text{for all} \quad x_\alpha, x \in P$$

$$x_\alpha \overset{(o)}{\to} x \Rightarrow x_\alpha \overset{\tau_o}{\to} x \qquad \text{for all} \quad x_\alpha, x \in P$$

$$x_\alpha \overset{(\hat{o})}{\to} x \Rightarrow x_\alpha \overset{\hat{\tau}_o}{\to} x \qquad \text{for all} \quad x_\alpha, x \in \hat{P}$$

Questions arise in connection with the opposite implications, namely, whether the following equivalences hold:

(o1) $\quad x_\alpha \overset{(o)}{\to} x \quad \text{iff} \quad x_\alpha \overset{(\hat{o})}{\to} x \qquad \text{for all} \quad x_\alpha, x \in P$

(o2) $\quad x_\alpha \overset{\tau_o}{\to} x \quad \text{iff} \quad x_\alpha \overset{\hat{\tau}_o}{\to} x \qquad \text{for all} \quad x_\alpha, x \in P$

(o3) $\quad x_\alpha \overset{(o)}{\to} x \quad \text{iff} \quad x_\alpha \overset{\tau_o}{\to} x \qquad \text{for all} \quad x_\alpha, x \in P$

(o4) $\quad x_\alpha \overset{(\hat{o})}{\to} x \quad \text{iff} \quad x_\alpha \overset{\tau_o}{\to} x \qquad \text{for all} \quad x_\alpha, x \in \hat{P}$

Some results are known for specific partially ordered structures as Boolean algebras and orthomodular posets.

If $P = B$ is a Boolean algebra then

(04) $\Rightarrow B$ is atomic (Erné, 1980)

B is atomic $\Rightarrow$ (o2) (Riecanová, 1993, Theorem 3.3 (iii))

B is separable $\Rightarrow$ (o1)–(o4) (Riecanová, 1997)

The same implications are true for an orthomodular lattice satisfying the following property (Riecanová, 1997):

Definition. An orthomodular lattice P is said to be *strongly compactly atomistic* if for every set M of atoms and every atom a with $a \leq b$ for every upper bound b of M there exists a finite set $F \subset M$ such that $a \leq \vee F$.

2. THE CONVERGENCE CLASS AXIOMS FOR THE ORDER CONVERGENCE

Pairs $(\{x_\alpha\}_{\alpha \in \mathscr{E}}, x)$ of nets and points are said to form a *convergence class* iff the following conditions are satisfied (Kelley, 1957):

(i) Every stationary net $\{x_\alpha\}_{\alpha\in\mathscr{E}}$, i.e., a net for which $x_\alpha = x$ for all $\alpha \in \mathscr{E}$, converges to x.

(ii) If a net $\{x_\alpha\}_{\alpha\in\mathscr{E}}$ converges to a point x, then every one of its subnets $\{x_\beta\}_{\beta\in\mathscr{F}}$ converges to x.

(iii) If for every $\alpha \in \mathscr{E}$ and every $\beta \in \mathscr{F}_\alpha$ a net $\{x_{\alpha,\beta}\}_{\beta\in\mathscr{F}_\alpha}$ converges to x_α and the net $\{x_\alpha\}_{\alpha\in\mathscr{E}}$ converges to x, then the net $\{x_{(\alpha,f(\alpha))}\}_{(\alpha,f)\in\mathscr{E}\times\Pi_{\alpha\in\mathscr{E}}\mathscr{F}_\alpha}$, where $\Pi_{\alpha\in\mathscr{E}}\mathscr{F}_\alpha$ is directed pointwise, converges to x.

(iv) If a net $\{x_\alpha\}_{\alpha\in\mathscr{E}}$ does not converge to a point x, then there exists a subnet $\{x_\beta\}_{\beta\in\mathscr{F}}$ no subnet of which converges to x.

According to Kelley (1957, Ch. 2, Th. 9), the order convergence coincides with the order topology convergence, i.e., equivalences (o3) and (o4) hold true if and only if the order convergence satisfies the conditions (i)–(iv) of the convergence class.

Directly from the definition of the order convergence it follows that the set of pairs of the order convergence nets and their limits in a poset P satisfy conditions (i) and (ii) of the convergence class.

The following example shows that the condition (iv) of the convergence class does not hold even though P is a complete lattice.

Example 1. Denote $N_j = \{a_{i,j} | i = 1, 2, \ldots\}$ for $j = 1, 2$ ordered naturally according to the first index and consider $(P, \leq)$ as a (0, 1)-pasting of the sets N_1, N_2.

It is easy to see that $(P, \leq)$ is a complete lattice. The sequence

$$\{a_{1,1},\ a_{1,2},\ a_{2,1},\ a_{2,2},\ a_{3,1},\ a_{3,2}\ \ldots\}$$

does not order converge in P. On the other hand, every subnet of this sequence contains a subsequence which order converges to 1.

The idea of the previous example is extended in the next one to show that condition (iii) of the convergence class may not be satisfied even in a complete lattice.

Example 2. Let us take P the (0, 1)-pasting of countably many countable sets $N_j = \{a_{i,j} | i = 1, 2, \ldots\}$, for $j = 1, 2, \ldots$, naturally ordered according to the first index.

P is a complete lattice. For every j the sequence $\{a_{i,j}\}_{i=1}^{\infty}$ converges to 1. The sequence $\{1\}_{j=1}^{\infty}$ is a stationary sequence converging to 1. However, the "diagonal" net $\{a_{f(j),j}\}_{j\in N, f\in N^N}$ does not order converge to 1.

3. THE ORDER TOPOLOGY IN A POSET AND ITS MACNEILLE COMPLETION

The equivalences (o1), (o2) concern the MacNeille completion $\hat{P}$ of P. Now, $(\hat{P}, \leq)$ is fully characterized as a complete lattice containing $(P, \leq)$, in which for every $x \in \hat{P}$ there are sets $M \subset P$ and $N \subset P$ such that $x = \vee M = \wedge N$. From the definition of order convergence it immediately follows that the order convergence in P implies the order convergence in $\hat{P}$. The example in Kirchheimová (1990) shows that the opposite implication is not true in general. However, the analogous problem remained open for order topologies. Again, it can be seen that the order topology in P is in general not weaker than the order topology in $\hat{P}$, restricted to P, but it was not clear whether they are not equal. The following example shows that they may differ, i.e., that $\hat{\tau}_o \cap P \neq \tau_o$.

Example 3. Let us denote

$$Q = \left\{ \left(n, \frac{1}{n} \right) | n = 1, 2, \ldots \right\}$$

$$P = Q \cup \{(n, -n) | n, = 0, 1, \ldots\} \cup \{(0, -\infty), (\infty, \infty)\}$$

and

$$M = P \cup \{(n, 0) | n = 1, 2, \ldots\}$$

Let M be partially ordered "by coordinates," i.e., $(a, b) \leq (c, d)$ iff $a \leq c$ and $b \leq d$. It is not difficult to verify that $(M, \leq)$ is a complete lattice.

Since every element in M can be expressed as a join and meet of elements of P, M is the MacNeille completion of P.

Since P contains finite chains only, the only (o)-convergent nets in P are (from a certain index) stationary nets, and the τ_o topology on P is the discrete one.

On the other hand, $(n, 0) \leq (n, 1/n) \leq (\infty, \infty)$, the sequence $\{(n, 0)\}_{n=1}^{\infty}$ is increasing, and $\wedge_{n=1}^{\infty} (n, 0) = (\infty, \infty)$. It follows that the sequence $\{(n, 0)\}_{n=1}^{\infty}$ (o)-converges to (∞, ∞) in M. Thus Q is not a τ_o-closed set in M and since $Q \cup \{(\infty, \infty)\} \subseteq P$, τ_o is strictly weaker than $\tau_o \cap P$.

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